Research Statement

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My research interests are in extremal and probabilistic combinatorics. My work so far encompasses several different problems under this broad umbrella, including graph saturation; a counting version of the Multidimensional Szemerédi Theorem; and thresholds for the anti-Ramsey problem. I am an expert in using several cutting-edge techniques, including the hypergraph container method, flag algebras and entropy, as well as being proficient in probabilistic methods and having a particular knack for finding intricate constructions.

I am currently a research fellow at the University of Warwick, working with Richard Montgomery. Prior to this, I was awarded a PIMS postdoctoral research fellowship to work at the University of Victoria with Natasha Morrison and Jonathan Noel. I also spent a year as a postdoctoral research fellow at Toronto Metropolitan University (formerly Ryerson University).

I am the sole author of two papers and I have co-authored a further fifteen papers with 20 different collaborators, including several prominent established researchers as well as fellow early-career mathematicians. My network has expanded rapidly in a relatively short time. I have been invited to give a total of fourteen external seminar talks at eleven different institutions as well as invited talks at the major international conferences MoCCA '20 and CanaDAM 2023. I have attended six prestigious and highly selective workshops, three by invitation and three by application, and will be attending a further three workshops this coming summer.

In this document, I begin in Section 1 with a brief discussion of a select few of my most significant research contributions, mentioning some avenues for future research along the way. In Section 2, I describe two future research projects in more detail. Finally, in Section 3, I discuss my plans for obtaining support for my research and outline some opportunities for collaboration.

1 Overview of significant research contributions

This section contains a selection of my work so far, highlighting directions for future research where relevant.

1.1 Graph saturation problems

Saturation problems aim to understand the tipping point at which a structure suddenly contains a particular pattern. For a fixed graph F we say that a graph G is *F-saturated* if it does not contain F as a subgraph, but adding any extra edge creates a copy of F as a subgraph. Erdős, Hajnal and Moon introduced the *saturation number* of F, denoted by $\mathrm{sat}(n, F)$, which is the minimum number of edges in an F-saturated graph G on n vertices. This is an opposite to the extremal Turán number $\mathrm{ex}(n, F)$ that counts the maximum number of edges in a maximal F-free (or equivalently, an F-saturated) graph on n vertices. The saturation number is in many ways less well behaved. For example, we know that the Turán density $\lim_{n\to\infty} \mathrm{ex}(n, F)/n^2$ always exists. However, although Kászonyi and Tuza [24] proved that in general $\mathrm{sat}(n, F) = O(n)$, Tuza's conjecture that $\mathrm{sat}(n, F)/n$ always tends to a limit remains open.

The definition of saturation extends naturally to families of graphs. Pikhurko [29] disproved a strengthening of Tuza's conjecture by finding a finite family $\mathcal{F}$ of graphs such that $\mathrm{sat}(n, \mathcal{F})/n$ does not converge as n tends to infinity. Pikhurko asked whether a similar behaviour can occur for families of r-uniform hypergraphs. I showed [5] that the answer to this question is yes. The proof is by construction of a family $\mathcal{F}$ (which grows in size with r). By adjusting this construction I also demonstrated that we can reduce the size of the forbidden family to contain just four hypergraphs, bringing it closer to Tuza's conjecture (which concerns single hypergraphs).

We can also generalise saturation to edge-coloured graphs, specifically *rainbow* colourings where every edge is assigned a distinct colour. An edge-coloured graph G is *F-rainbow saturated* if it does not contain a rainbow copy of F, but the addition of any non-edge in any colour creates a rainbow copy of F. The *rainbow saturation number* of F, denoted by $\mathrm{rsat}(n, F)$, is then the minimum number of edges in an F-rainbow-saturated edge-coloured graph on n vertices. Girão, Lewis, and Popielarz [21] conjectured that, like the usual saturation number, the rainbow saturation number of any non-empty graph F is at most linear in n. We proved this conjecture [9]. In addition, we found an improved upper bound on the rainbow saturation number of the complete graph, disproving another conjecture of Girão, Lewis, and Popielarz.

Since we have $\mathrm{rsat}(n, F) = O(n)$ for all F, it is natural to make a conjecture analogous to Tuza's, and to ask whether the limit $\lim_{n\to\infty} \mathrm{rsat}(n, F)/n$ exists for all graphs F. This is a promising direction for future research: the question is neither strictly weaker or stronger than Tuza's conjecture, but it is certainly less thoroughly studied. Various results on the saturation number look likely to be translatable to the rainbow setting and proving the corresponding results may shed light on the area more generally. For example, what if F is replaced by a finite family of graphs, as in Pikhurko's result [29]? What about hypergraphs? I intend to combine my work on these two areas of saturation to answer such questions.

1.2 A counting version of the Multidimensional Szemerédi Theorem

Szemerédi's Theorem [32] states that the maximum cardinality of a subset of $[n] := \{1, \dots, n\}$ not containing k points in an arithmetic progression is $o(n)$, settling a conjecture of Erdős and Turán from the 1930s. Szemerédi's original proof led to the discovery of his celebrated Regularity Lemma.

The Multidimensional Szemerédi Theorem (Theorem 1.1 below), first proven by Furstenberg and Katznelson [20] is a natural generalisation. Given $d \in \mathbb{N}$ and a set $X \subseteq \mathbb{N}^d$, a *non-trivial copy* of X is a set of the form

$$\vec{b} + rX := \{\vec{b} + r\vec{x} : \vec{x} \in X\},$$

where $\vec{b} \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$. Let $r_X(n)$ denote the cardinality of the largest subset of $[n]^d$ which does not contain a non-trivial copy of X; such a set is said to be *X-free*. For $d = 1$, we simply write $r_k(n)$ to mean $r_{\{1,\dots,k\}}(n)$. In this language, Szemerédi's Theorem says that $r_k(n) = o(n)$, for any $k \geq 3$.

Theorem 1.1 (Multidimensional Szemerédi Theorem [20])**.** *If $d \geq 1$ and X is a finite subset of $\mathbb{N}^d$, then $r_X(n) = o(n^d)$.*

We focused on the related question of counting the number of X-free subsets of $[n]^d$. It is trivial to see that there are at least $2^{r_X(n)}$ such sets. A well-known (and still open) question of Cameron and Erdős [16] asks whether this bound is approximately correct for k-term arithmetic progressions. Balogh, Liu, and Sharifzadeh [3] made partial progress towards this question by showing that for infinitely many n the number of k-term-arithmetic-progression-free subsets of $[n]$ is $2^{O(r_k(n))}$. We proved the following generalisation to the multidimensional setting.

Theorem 1.2 ([8])**.** *Let d be a positive integer and let $X \subseteq \mathbb{N}^d$ be a finite set such that $|X| \geq 3$. For infinitely many $n \in \mathbb{N}$, the number of X-free subsets of $[n]^d$ is $2^{O(r_X(n))}$.*

As in [3], we obtained this result by applying the seminal hypergraph container method, a widely applicable tool which has been used to prove a myriad of enumeration results.

1.3 Thresholds for constrained Ramsey and anti-Ramsey problems

For fixed graphs H_1, H_2, we say that a graph G has the *constrained Ramsey property for* (H_1, H_2), denoted $G \xrightarrow{\text{c-ram}} (H_1, H_2)$, if any edge-colouring of G contains either a monochromatic copy of H_1 or a rainbow copy of H_2, i.e. a copy of H_2 where each edge has a different colour. It is not hard to see that G cannot have the constrained Ramsey property unless either H_1 is a star or H_2 is a forest.

We wish to know when the *random graph* $G(n, p)$ has the constrained Ramsey property for (H_1, H_2) with high probability. Precisely, we wish to find a threshold function $f : \mathbb{N} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{c-ram}} (H_1, H_2)] = \begin{cases} 0 & \text{if } p = o(f(n)) \\ 1 & \text{if } p = \omega(f(n)). \end{cases}$$

We will use the term *0-statement* to refer to the statement for $p = o(f(n))$, and *1-statement* to refer to the statement for $p = \omega(f(n))$.

Question 1.3 (Constrained Ramsey)**.** *Let H_1, H_2 be graphs such that H_1 is a star or H_2 is a forest. What is a threshold function for the constrained Ramsey property for (H_1, H_2) in $G(n, p)$?*

My main result [7] gives a 0-statement for the constrained Ramsey property in $G(n, p)$ whenever H_1 is a star on ≥ 3 edges and H_2 is not a forest. This matches a corresponding 1-statement given in [26]. The remaining non-trivial cases (when H_2 is a forest) have 0-statement covered by the Rödl–Ruciński random Ramsey theorem [30] and corresponding 1-statement given in [18]. In particular, our theorem thus resolves question 1.3 for all non-trivial cases with the exception of $H_1 = K_{1,2}$, which is equivalent to the anti-Ramsey property for H_2.

For a fixed graph H, we say that G has the *anti-Ramsey property for H* if any proper edge-colouring of G contains a rainbow copy of H. The study of the anti-Ramsey problem was initiated by Rödl and Tuza [31] who focused on the case when H is a cycle. As with the constrained Ramsey problem, we consider the question of finding a threshold for $G(n, p)$ having the anti-Ramsey property for a given H.

For the anti-Ramsey problem, we have some partial progress, reducing proving the 0-statement to a (necessary) colouring statement. We proved [7] the colouring statement for some special cases of H, including when H is a d-regular graph on at least $4d$ vertices. However, the anti-Ramsey case of Question 1.3 remains open in general.

In [28] there is a general framework for proving 0-statements in various settings, which is applied to show that the 0-statement for anti-Ramsey holds for sufficiently long cycles and sufficiently large complete graphs. We use a specially adapted version of this framework to prove our results.

1.4 Subgraph games in the semi-random graph process

The semi-random graph process is a single-player game that begins with an empty graph on n vertices. In each round, a vertex u is presented to the player independently and uniformly at random. The player then adaptively selects a vertex v and adds the edge uv to the graph. For a fixed monotone graph property, the objective of the player is to force the graph to satisfy this property with high probability in as few rounds as possible.

We considered the problem where the player is aiming to construct a subgraph isomorphic to an arbitrary fixed graph G. Let $\omega = \omega(n)$ be any function tending to infinity with n. In [15], it was proved that asymptotically almost surely one can construct G in less than $n^{(d-1)/d}\omega$ rounds where $d \geq 2$ is the degeneracy of G. Moreover, they conjectured that this upper bound is sharp for all graphs G and proved sharpness when G is complete. With Trent Marbach, Paweł Prałat, and Andrzej Ruciński, I proved this conjecture in full generality [14].

There is a natural generalization of this to s-uniform hypergraphs, in which r vertices are chosen at random, and the player selects $s - r$ vertices. Our results easily generalise to hypergraphs when $r = 1$, but when $r \geq 2$, thresholds are not known even for complete hypergraphs. We found improved upper and lower bounds [13].

2 Future research activities

I will discuss in more detail my plans for solving some fascinating open problems in two different research areas.

2.1 Common pairs of graphs

Ramsey theory tells us that in any sufficiently large structure, particular patterns will emerge somewhere. This is exemplified by Ramsey's foundational theorem that for every graph H, any sufficiently large red/blue edge-coloured complete graph must contain a monochromatic copy of H. This project studies the minimum density of those patterns, and specifically, it considers the closely related *Ramsey multiplicity problem* which asks for the asymptotics of the minimum possible number of monochromatic copies of H in a red/blue edge-colouring of the complete graph K_n as n tends to infinity.

A beautiful result of Goodman [22] found the Ramsey multiplicity of the triangle by proving that the number of monochromatic triangles is minimized by an unbiased random edge colouring. Inspired by this, a graph H is called *common* if, like the triangle, the number of monochromatic labelled copies of H in a red/blue edge colouring of a large complete graph is asymptotically minimized by an unbiased random colouring. Common graphs are intimately linked to Sidorenko's famous conjecture on the homomorphism density of bipartite graphs.

In the asymmetric setting, a pair of graphs (H_1, H_2) is called $(p, 1-p)$-common if the minimum of the appropriately weighted sum of densities of red copies of H_1 and blue copies of H_2 is attained by a random colouring where each edge is coloured red with probability p and blue with probability $1 - p$, as studied in my recent work [11, 12]. Our results include off-diagonal extensions of several standard theorems on common graphs and novel results for common pairs of graphs with no natural analogue in the classical setting. Further, we found a significant new family of graphs that are common, both in pairs and in the usual sense, using the information-theoretic tool of entropy. This paper has already generated a lot of additional work on the area [25, 34, 17].

Arising from this work is a natural surprisingly non-trivial conjecture, which is my first objective.

Conjecture 2.1 (Conjecture 1.2 in [11])**.** *If H_1 and H_2 are graphs then the set $\pi(H_1, H_2)$ defined to be $\{p \in (0, 1) : (H_1, H_2)$ is $(p, 1-p)$-common$\}$ is an interval.*

In my work [11], we find examples of pairs of graphs for which the set $\pi(H_1, H_2)$ is empty, the entire interval $(0, 1)$, or a single point. Notably, we use the method of flag algebras to prove that $\pi(C_4, C_5)$ is a proper subset of $(0, 1)$ containing at least two elements.

Thomason [33] proved that K_4 is not common, which was extended by Jagger, Šťovíček and Thomason [23] to the very general statement that any graph containing K_4 is not common. In [11], my co-authors and I proved that both graphs in any $(p, 1-p)$-common pair (H_1, H_2) must be K_4-free. My second objective is to determine whether the existence of a K_4 subgraph is in some sense 'the only barrier' to being common. Specifically, I aim to settle the following two conjectures:

Conjecture 2.2 ([11])**.** *For any K_4-free graph F, there exists a connected common graph H containing F as a subgraph. There is also a connected common graph H containing F as an induced subgraph.*

Conjecture 2.3 ([11])**.** *If H_1 is K_4-free, there is a graph H_2 and $p \in (0, 1)$ such that (H_1, H_2) is $(p, 1-p)$-common.*

It would also be interesting to prove a weaker version of Conjecture 2.3 restricted to bipartite graphs, which could be viewed as a natural weak form of Sidorenko's Conjecture.

Any progress towards a problem as famous as Sidorenko's conjecture would be of high interest to researchers in combinatorics and beyond and is likely to be published in a top journal. In addition, methods developed to solve these problems are likely to have applications to the many other compelling graph homomorphism density questions.

The first step to solving these kinds of problem is often to translate the problem into one of *graph limits*, one benefit of which is that the smaller order asymptotic terms necessary in the setting of finite graphs vanish in the limit. This analytic approach lets us re-parameterise to obtain equivalent algebraic expansions that are easier to work with, as used effectively in eg. [23]. The development of graph limits was one of many accomplishments cited in the award of the 2021 Abel prize to Lovász.

I intend to prove the commonality of certain graphs by imitating the recent novel application of *Schur convexity*, which was used to extend the commonality of paths and cycles [25]. Then, certain types of binomial inequalities for homomorphism densities allow us to conclude that if one pair of graphs is $(p, 1-p)$-common then so are many others, as in our paper [12], which enables us to extrapolate much further. Such inequalities can be proved using the information-theoretic notion of *entropy*, which was used to make substantial progress towards Sidorenko's conjecture [19]. In addition to the aforementioned use of entropy in [12], I also used entropy to great effect to find Sidorenko-type inequalities for pairs of trees [6].

For Conjecture 2.2 specifically, a natural approach is to consider a so-called *blow-up* of the graph F (as used similarly in [27]). The presence of large complete bipartite subgraphs in a graph tends to inflate the homomorphism density, and a blow-up by construction has many such subgraphs.

Finally, if any particular graph comes up as an obstacle to proving the conjectures using the above techniques, I can verify conjectured counterexamples using the *flag algebra method*, for which Razborov won the 2013 David P. Robbins Prize from the AMS. The idea is that it suffices to solve a certain semi-definite relaxation of the problem, which can be solved approximately by computer using standard programming packages. An exact solution is obtained by 'rounding' the approximate one.

2.2 The invertibility of oriented graphs

For an oriented graph D and a set $X \subseteq V(D)$, the inversion of X in D is the graph obtained from D by reversing the orientation of each edge that has both endpoints in X. Define the inversion number of D, denoted $\text{inv}(D)$, to be the minimum number of inversions required to obtain an acyclic oriented graph from D.

A conjecture of [4] stated that for any oriented graphs D_1 and D_2, we have $\text{inv}(D_1 \to D_2) = \text{inv}(D_1) + \text{inv}(D_2)$ where $D_1 \to D_2$ is the *dijoin*, the oriented graph constructed from vertex-disjoint copies of D_1 and D_2 by adding all edges uv where $u \in V(D_1)$ and $v \in V(D_2)$. This natural conjecture was surprisingly disproved simultaneously by two groups [1, 2] who proved that there exist digraphs D_1 with $\text{inv}(D_1 \to D_2) = \text{inv}(D_1) + \text{inv}(D_2) - 1$, where interestingly all of the counterexamples require $\text{inv}(D_1)$ and $\text{inv}(D_2)$ to be odd. This begs the question: what is the true lower bound of $\text{inv}(D_1 \to D_2)$?

We proved [10] that $\text{inv}(D_1 \to D_2) > \text{inv}(D_1)$, for any (non-acyclic) oriented graphs D_1 and D_2 such that $\text{inv}(D_1) = \text{inv}(D_2)$, answering a question from [2]. To do so, we introduced some more general tools and theory, finding a correspondence with the rank of certain matrices over $\mathbb{F}_2$, which gives an explanation for the counterexamples depending on parity. Another paper [35], released almost simultaneously, proved that if $\text{inv}(D)$ is even then $\text{inv}(D \to \overrightarrow{K_3}) = \text{inv}(D) + \text{inv}\left(\overrightarrow{K_3}\right)$ where $\overrightarrow{K_3}$ is the oriented 3-cycle. Combining the approaches of these two papers will be the first step towards an answer to the question.

There are a number of other open problems on the inversion number, see for example [1, 2], and this would therefore be a particularly suitable project to involve a research student in, as we are likely to discover many interesting related results along the way.

3 Research support and collaboration

To build and maintain a successful research career it is critical to secure funding to support students, postdoctoral researchers, travel, and visits from collaborators. If appointed I intend to apply for an EPSRC New Investigator Award, an EPSRC Early Career Fellowship, a Royal Society University Research Fellowship and a Royal Society Research Grant. These sources of funding will boost my research further and I will later apply for more substantial funding sources including, for example, an EPSRC Standard Grant and a Leverhulme Trust Research Project Grant. In addition I will continue to actively seek support to fund conferences and workshops. My past successes in procuring a PIMS Postdoctoral Research Fellowship and various sources of travel funding indicates my potential for success in these future grant applications.

References

[1] N. Alon, E. Powierski, M. Savery, A. Scott, and E. Wilmer. "Invertibility of digraphs and tournaments". In: *SIAM J. Discrete Math.* 38.1 (2024).

[2] G. Aubian, F. Havet, F. Hörsch, F. Klingelhoefer, N. Nisse, C. Rambaud, and Q. Vermande. "Problems, proofs, and disproofs on the inversion number". arXiv: 2212.09188.

[3] J. Balogh, H. Liu, and M. Sharifzadeh. "The number of subsets of integers with no k-term arithmetic progression". In: *Int. Math. Res. Not. IMRN* 20 (2017).

[4] J. Bang-Jensen, J. Costa Ferreira da Silva, and F. Havet. "On the inversion number of oriented graphs". In: *Discrete Math. Theor. Comput. Sci.* 23.2 (2023).

[5] N. Behague. "Hypergraph saturation irregularities". In: *Electron. J. Comb.* 25.2 (2018). (13 pages).

[6] N. Behague, G. Crudele, J. A. Noel, and L. M. Simbaqueba. "Sidorenko-Type Inequalities for Pairs of Trees". arXiv: 2305.16542.

[7] N. Behague, R. Hancock, J. Hyde, S. Letzter, and N. Morrison. "Thresholds for constrained Ramsey and anti-Ramsey problems". In: *Eur. J. Combin.* (2024). (accepted) (27 pages). arXiv: 2401.06881.

[8] N. Behague, J. Hyde, N. Morrison, J. A. Noel, and A. Wright. "An Approximate Counting Version of the Multidimensional Szemerédi Theorem". (19 pages). 2023. arXiv: 2311.13709.

[9] N. Behague, T. Johnston, S. Letzter, N. Morrison, and S. Ogden. "The Rainbow Saturation Number Is Linear". In: *SIAM J. Discrete Math.* 38.2 (2024). (11 pages).

[10] N. Behague, T. Johnston, N. Morrison, and S. Ogden. "A note on the invertibility of oriented graphs". (10 pages). 2024. arXiv: 2404.10663.

[11] N. Behague, N. Morrison, and J. A. Noel. "Common Pairs of Graphs". (27 pages). 2023. arXiv: 2208.02045.

[12] N. Behague, N. Morrison, and J. A. Noel. "Off-Diagonal Commonality of Graphs via Entropy". In: *SIAM J. Discrete Math.* 38.3 (2024). (26 pages).

[13] N. Behague, P. Prałat, and A. Ruciński. "Creating Subgraphs in Semi-Random Hypergraph Games". (39 pages). 2024. arXiv: ..

[14] N. C. Behague, T. G. Marbach, P. Prałat, and A. Ruciński. "Subgraph Games in the Semi-Random Graph Process and Its Generalization to Hypergraphs". In: *Electron. J. Comb.* 31.3 (2024). (18 pages).

[15] O. Ben-Eliezer, D. Hefetz, G. Kronenberg, O. Parczyk, C. Shikhelman, and M. Stojaković. "Semi-random graph process". In: *Random Structures & Algorithms* 56.3 (2020).

[16] P. J. Cameron and P. Erdős. "On the number of sets of integers with various properties". In: *Number theory (Banff, AB, 1988)*. de Gruyter, Berlin, 1990.

[17] H. Chen and J. Ma. "A Property on Monochromatic Copies of Graphs Containing a Triangle". In: *SIAM Journal on Discrete Mathematics* 38.1 (2024). eprint: https://doi.org/10.1137/23M1564894.

[18] M. Collares, Y. Kohayakawa, C. G. Moreira, and G. O. Mota. "The threshold for the constrained Ramsey property". arXiv: 2207.05201.

[19] D. Conlon, J. H. Kim, C. Lee, and J. Lee. "Some advances on Sidorenko's conjecture". In: *J. Lond. Math. Soc. (2)* 98.3 (2018).

[20] H. Furstenberg and Y. Katznelson. "An ergodic Szemerédi theorem for commuting transformations". In: *J. Analyse Math.* 34 (1978).

[21] A. Girão, D. Lewis, and K. Popielarz. "Rainbow saturation of graphs". In: *J. Graph Theory* 94.3 (2020).

[22] A. W. Goodman. "On sets of acquaintances and strangers at any party". In: *Amer. Math. Monthly* 66 (1959).

[23] C. Jagger, P. Šťovíček, and A. Thomason. "Multiplicities of subgraphs". In: *Combinatorica* 16.1 (1996).

[24] L. Kászonyi and Z. Tuza. "Saturated graphs with minimal number of edges". In: *J. Graph Theory* 10.2 (1986).

[25] J. S. Kim and J. Lee. "Extended commonality of paths and cycles via Schur convexity". In: *Journal of Combinatorial Theory, Series B* 166 (2024).

[26] Y. Kohayakawa, P. B. Konstadinidis, and G. O. Mota. "On an anti-Ramsey threshold for random graphs". In: *European J. Combin.* 40 (2014).

[27] D. Králˇ, J. A. Noel, S. Norin, J. Volec, and F. Wei. "Non-bipartite k-common graphs". In: *Combinatorica* 42.1 (2022).

[28] R. Nenadov, Y. Person, N. Škorić, and A. Steger. "An algorithmic framework for obtaining lower bounds for random Ramsey problems". In: *J. Combin. Theory Ser. B* 124 (2017).

[29] O. Pikhurko. "Results and open problems on minimum saturated hypergraphs". In: *Ars Combin.* 72 (2004).

[30] V. Rödl and A. Ruciński. "Threshold Functions for Ramsey Properties". In: *J. Amer. Math. Soc.* 8.4 (1995).

[31] V. Rödl and Z. Tuza. "Rainbow subgraphs in properly edge-colored graphs". In: *Random Structures Algorithms* 3.2 (1992).

[32] E. Szemerédi. "On sets of integers containing no k elements in arithmetic progression". In: *Acta Arith.* 27 (1975).

[33] A. Thomason. "A disproof of a conjecture of Erdős in Ramsey theory". In: *J. London Math. Soc. (2)* 39.2 (1989).

[34] L. Versteegen. "Strongly common graphs with odd girth are cycles". arXiv: 2305.10903.

[35] H. Wang, Y. Yang, and M. Lu. "The inversion number of dijoins and blow-up digraphs". arXiv: 2404.14937.