Research Statement

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November 30, 2020

My research interests are in extremal and probabilistic combinatorics. My work so far encompasses several different problems, all variations on the usual extremal combinatorics theme of looking at the existence of certain substructures or properties of a family of combinatorial objects (graphs, hypergraphs, automata) as the objects grow large. A particular pattern that emerges from my work is a use of intricate constructions, which occur not only as counterexamples, but are also used to show positively that various things are possible and as tools to bound probabilities. Each section below discusses in more depth a different problem I have worked on.

Section 1. *Hypergraph saturation irregularities.* Tuza's famous conjecture on the saturation number states that for r-uniform hypergraphs F the value $\frac{\mathrm{sat}(F,n)}{n^{r-1}}$ converges. I answered a question of Pikhurko concerning the asymptotics of the saturation number for families of hypergraphs, proving in particular that $\frac{\mathrm{sat}(\mathcal{F},n)}{n^{r-1}}$ need not converge if $\mathcal{F}$ is a family of r-uniform hypergraphs.

Section 2. *Synchronizing automata and Černý's conjecture.* An automaton on n states is synchronizing if there is a word that maps all n states onto the same state. Černý's conjecture on the length of the shortest such word is probably the most famous open problem in automata theory. With Robert Johnson, I considered the closely related question of determining the minimum length $\mathrm{rdv}(k, n)$ of a word that maps k states onto a single state. We improved the upper bound on $\mathrm{rdv}(3, n)$ from $0.5n^2$ to $\approx 0.19n^2$ and further extended this to improved upper bounds on $\mathrm{rdv}(4, n)$ and $\mathrm{rdv}(5, n)$.

Section 3. *Semi-perfect 1-factorizations of the hypercube.* The existence or non-existence of perfect 1-factorizations has been studied for various families of graphs, with perhaps the most famous open problem in the area being Kotzig's conjecture which states that the complete graph K_{2n} has a perfect 1-factorization. I focused on another well-studied family of graphs: the hypercubes Q_d. I answered almost fully the question of how close (in some particular sense) to perfect a 1-factorization of the hypercube can be.

Section 4. *Connectivity of high dimension k-nearest neighbour graphs.* The k-nearest neighbour random geometric graph model puts vertices randomly in a d-dimensional box and joins each vertex to its k nearest neighbours. I found significantly improved upper and lower bounds on the threshold for connectivity for the k-nearest neighbour graph in high dimensions.

I will finish with a brief discussion of some directions for further work.

1 Hypergraph saturation irregularities

Given a fixed graph F and a positive integer n, Turán's extremal number, $\mathrm{ex}(F, n)$, is the maximal number of edges an F-free graph G on n vertices can have (see figure 1). What if instead we were to ask about the minimal number of edges? If we simply replace 'maximal' with 'minimal' in the definition above we get a trivial function, since a graph with no edges will be F-free.

Notice that in any maximal F-free graph it must be the case that adding any edge creates a copy of F. This inspires the following definition: we say a graph G is *F-saturated* if it does not contain a copy of F as a subgraph but adding any edge creates a copy of F. Now we have an equivalent definition of Turán's extremal number:

$$\text{ex}(F, n) = \max\{e(G) : |G| = n \text{ and } G \text{ is } F\text{-saturated}\}$$

and if we replace max with min we get the *saturation number*:

$$\text{sat}(F, n) = \min\{e(G) : |G| = n \text{ and } G \text{ is } F\text{-saturated}\}.$$

Turán's Theorem and the Erdős-Stone Theorem tell us that, for a (non-empty) graph F, $\text{ex}(F, n) = \left(1 - \frac{1}{\chi(F)-1} + o(1)\right)\binom{n}{2}$, where $\chi(F)$ is the chromatic number of F. In particular, $\text{ex}(F, n)/\binom{n}{2}$ converges to a limit as n tends to infinity.

Can we say anything similar about the saturation number? Kászonyi and Tuza [KT86] proved that $\text{sat}(F, n) = O(n)$. Tuza [Tuz88] went on to conjecture that for every graph F the limit $\lim_{n\to\infty} \frac{\text{sat}(F,n)}{n}$ exists.

The notion of saturation can be generalised to families of graphs. For a family $\mathcal{F}$ of graphs (called a forbidden family), an graph H is called $\mathcal{F}$*-saturated* if it does not contain any graph in $\mathcal{F}$ as a subgraph, but adding any edge creates a copy of some graph $F \in \mathcal{F}$ as a subgraph of H. We define the saturation number in the same way as before:

$$\text{sat}(\mathcal{F}, n) = min\{e(H) : |H| = n \text{ and } H \text{ is } \mathcal{F}\text{-saturated}\}.$$

For a family $\mathcal{F}$ of graphs we have $\text{sat}(\mathcal{F}, n) = O(n)$ [KT86], just as we did for single graphs. However, the generalisation of Tuza's conjecture to finite families of graphs is not true: an example of a finite family $\mathcal{F}$ where $\text{sat}(\mathcal{F}, n)/n$ does not tend to a limit was given by Pikhurko [Pik04], who then asked whether a similar construction was possible for families of r-uniform hypergraphs.

The definitions of saturation and saturation numbers transfer immediately to the context of r-uniform hypergraphs, which I will call r*-graphs*. For a family $\mathcal{F}$ of r-raphs, it was shown by Pikhurko [Pik99] that $\text{sat}(\mathcal{F}, n) = O\left(n^{r-1}\right)$ when the family contains only a finite number of graphs. This leads to the following generalisation of Tuza's conjecture to r-graphs, first posed by Pikhurko [Pik99].

Conjecture 1.1 (Tuza's Conjecture Generalised to r-graphs)**.** *For every r-graph F the limit $\lim_{n\to\infty} \frac{sat(F,n)}{n^{r-1}}$ exists.*

As in the 2-graph case we can further generalise this conjecture by replacing the single r-graph F with a finite family of r-graphs $\mathcal{F}$. Pikhurko asked whether there exists a family $\mathcal{F}$ of r-graphs such that $\text{sat}(\mathcal{F}, n)/n^{r-1}$ does not converge to a limit as n tends to infinity (problem 7 in [Pik04]). I have shown that the answer to this question is yes.

Theorem 1.2 ([Beh18])**.** *For all $r \geq 2$ there exists a family $\mathcal{F}$ of r-graphs and a constant $k \in \mathbb{N}$ such that*

$$sat(\mathcal{F}, n) = \begin{cases} O(n) & \text{if } k \mid n \\ \Omega\left(n^{r-1}\right) & \text{if } k \nmid n \end{cases}$$

In particular, $\frac{sat(\mathcal{F},n)}{n^{r-1}}$ does not converge.

The proof uses a similar approach to Pikhurko's construction for graphs. However, in the hypergraph case the construction is more complex and proving that it has the desired property is also a more difficult problem. We choose a constant k and to define a forbidden family $\mathcal{F}$ such that when k divides n there is a 'nice' construction of an $\mathcal{F}$-saturated graph with few edges, and when k does not divide n an $\mathcal{F}$-saturated graph requires comparatively many edges.

The forbidden family used contains two types of r-graph: a pair of intersecting complete r-graphs; or some disjoint complete r-graphs together with a 'bridge' edge between them. An

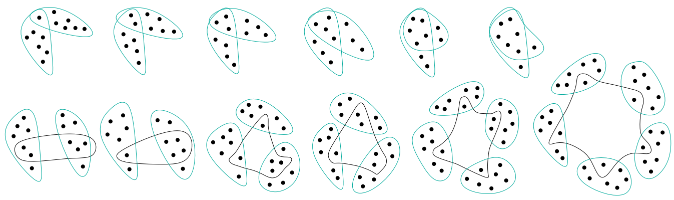

Figure 1: The family $\mathcal{F}$ of r-graphs from Theorem 1.2 for $r = 5$ and $k = 7$

example can be seen in figure 1, where the vertices surrounded by a blue line represent a copy of $K_k^{(r)}$ and vertices grouped by a black line represent a bridge edge.

The proof of Theorem 1.2 uses a family $\mathcal{F}$ which grows in size with r. By a different construction we show that we can reduce the size of the forbidden family to be a constant independent of r, bringing it closer to Tuza's conjecture which concerns single r-graphs.

Theorem 1.3 ([Beh18])**.** *For all $r \geq 3$ there exists a family $\mathcal{F}$ of four r-graphs such that $\frac{sat(\mathcal{F},n)}{n^{r-1}}$ does not converge.*

2 Synchronizing automata and Černý's conjecture

This is joint work with Robert Johnson.

A (deterministic, finite) automaton consists of a finite set of states – usually $[n]$ – and a finite set of *transition functions*, which are functions from the set of states to itself. We are interested in the results of applying a sequence of transitions to the set of states – we call such a sequence of transitions a *word* of the automaton. We say that a word is a *reset word* if it sends every state to the same point, and we call an automaton *synchronizing* if it has a reset word.

Conjecture 2.1 (Černý's Conjecture)**.** *Suppose an automaton on n states is synchronizing. Then there exists a reset word of length at most $(n-1)^2$.*

This conjecture comes from a particular family of automata, shown in Figure 2 for the $n = 4$ case, which have shortest reset word length $(n-1)^2$. Thus if Černý's conjecture were true it would be best possible.

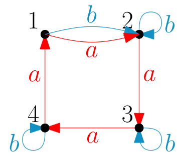

Figure 2: The Černý automaton for $n = 4$.

Černý's conjecture has been shown to hold for certain classes of automata, including orientable automata [Epp90], automata where one transition function is a cyclic permutation of the states [Dub98], and automata where the underlying digraph is Eulerian [Kar03]. For a survey of these and other results see [Vol08]. It remains open to prove the conjecture for all automata.

One can easily obtain a naive upper bound on the length of a shortest reset word by observing that for any given pair of states the shortest path from them to a single state will never pass through the same pair of states twice, and so there is a word of length $\leq \binom{n}{2}$ sending them to a single state. Applying this repeatedly gives a reset word of length at most $(n-1)\binom{n}{2}$.

An improved upper bound for the length of a minimal reset word comes from a result due to Frankl and Pin [Fra82] [Pin83]. Rather than only considering the shortest word sending a given pair to a singleton, this instead bounds the length of the shortest word sending a given k-set to a $(k-1)$-set.

Theorem 2.2 (Frankl–Pin)**.** *Consider a synchronizing automaton with state set Ω of size n. Let $S \subseteq \Omega$ be a set of size k where $k \geq 2$. There exists a word w of length at most $\binom{n-k+2}{2}$ such that $|w(S)| < k$.*

Applying Theorem 2.2 repeatedly, we get

Corollary 2.3 (Frankl–Pin)**.** *An n-state synchronizing automaton has a reset word of length $\leq \sum_{i=2}^{n} \binom{n-i+2}{2} = \frac{n^3-n}{6}$.*

This was the best known upper bound until relatively recently, and still only slight improvements to the constant factor have now been found: the atate of the art is $\approx 0.1654n^3 + o\left(n^3\right)$ due to Shitov [Shi19] building on work of Szykuła [Szy18].

A natural generalisation of Černý's question is to determine the length of the shortest path taking a k-set to a singleton. Specifically, what is the minimal value of $\mathrm{rdv}(k,n)$ such for any synchronizing automaton Ω on $[n]$ there is some k-set S and some word w of length at most $\mathrm{rdv}(k,n)$ such that $w(S)$ is a singleton? The function $\mathrm{rdv}(k,n)$ was introduced by Gonze and Jungers [GJ16] as the k-set rendezvous time.

Theorem 2.2 gives $\mathrm{rdv}(k,n) \leq 1 + \sum_{i=2}^{k-1} \binom{n-i+2}{2} = \frac{k-2}{2}n^2 + O(n)$. We have a straightforward argument almost halving this to $\left\lfloor \frac{k-1}{2} \right\rfloor \frac{n^2}{2}$.

However, our main result is an improved upper bound for the triple rendezvous time.

Theorem 2.4 ([BJ20])**.** *For all $n \geq 3$, we have $\mathrm{rdv}(3,n) \leq \frac{3-\sqrt{5}}{4}n^2 + \frac{3}{2}n$.*

This appears to be the first improvement over the trivial $\binom{n}{2}$ bound (there is a claimed bound of $n^2/4$ in [GJ16] but we believe this to be incorrect). We can also extend the techniques used on $\mathrm{rdv}(3,n)$ to further improve the upper bound for $\mathrm{rdv}(4,n)$ and $\mathrm{rdv}(5,n)$.

In a different direction, we examined the rendezvous time for automata that are not synchronizing but which have some k-set that can be reset. What is the minimal value of $\mathrm{rdv}^*(k,n)$ such for any automaton Ω on $[n]$ that has a k-set that an be reset has some k-set that can be reset in $\mathrm{rdv}^*(k,n)$ steps?

A naive upper bound on $\mathrm{rdv}^*(k,n)$ is $1 + \sum_{i=2}^{k-1} \binom{n}{i}$, since a shortest word down to a singleton will take a set through each set of size $< k$ at most once. We can show that this is, surprisingly, essentially best possible — that is, if k is fixed then $\mathrm{rdv}^*(k,n) = \Theta\left(n^{k-1}\right)$. The non-synchronizing case therefore exhibits very different behaviour to the synchronizing case, which implies that any approach to Černý's conjecture using rendezvous times must use the condition that the automata is synchronizing in a critical way.

3 Semi-perfect 1-factorizations of the hypercube

A 1-factorization of a graph H is a partition of the edges of H into disjoint perfect matchings $\{M_1, M_2, \ldots, M_n\}$, also known as 1-factors.

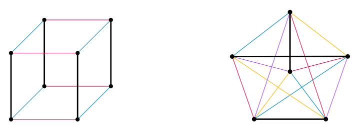

Figure 3: Examples of 1-factorizations of Q_3 and K_6

Note that the union of two 1-factors is a collection of disjoint cycles. We say that a 1-factorization is *perfect* if the union of any pair of distinct 1-factors is a Hamilton cycle. For example, in figure 3 the 1-factorization of Q_3 is not perfect but the 1-factorization of K_6 is perfect.

Kotzig [Kot64] conjectured that the complete graph K_{2n} has a perfect 1-factorization for all $n \geq 2$. This has long been outstanding and has so far only been shown to hold for n prime and

$2n-1$ prime (independently by Anderson and Nakamura [And73; Nak75]), as well as certain other small values of n (see [Wal97] for references).

The existence or non-existence of perfect 1-factorizations has been studied for various other families of graphs, in particular for the d-dimensional hypercube Q_d. It is easy to find a 1-factorization of the hypercube: take the i^{th} 1-factor to be all edges in direction i — we call this the *directional 1-factorization.* Note that the union of any pair of directional 1-factors is a disjoint union of 4-cycles.

It turns out there is no perfect 1-factorization of the hypercube.

Theorem 3.1. *[Lau80] Let H be a regular bipartite graph of degree d on vertex classes X and Y of size n. If n is even and $d > 2$ then H has no perfect 1-factorization.*

Since the hypercube is bipartite with vertex classes the odd and the even vertices, a corollary of this theorem is that for $d > 2$ there is no perfect 1-factorization of Q_d.

We can say more however by analysing the proof of theorem 3.1. In doing so, we discover that for a 1-factorization $\mathcal{M} = \{M_1, \dots, M_d\}$ of the hypercube Q_d there can be no odd-length cycle of matchings M_{i_1}, $M_{i_2}, \dots$, M_{i_k}, M_{i_1} with each consecutive pair forming a Hamilton cycle.

Define an auxiliary graph $G[\mathcal{M}]$ with vertices labelled $M_1, \dots, M_d$ and an edge from M_i to M_j if $M_i \cup M_j$ is a Hamilton cycle in H. Note that $\mathcal{M}$ is a perfect 1-factorization if and only if $G[\mathcal{M}]$ is a complete graph. Theorem 3.1 says that $G[\mathcal{M}]$ cannot be complete, and moreover by our observation above we have that $G[\mathcal{M}]$ does not contain any odd cycles. In particular, $G[\mathcal{M}]$ must be bipartite.

In the light of this observation, the only remaining question is whether for any k, l there is a 1-factorization $\mathcal{M}$ of Q_{k+l} such that $G[\mathcal{M}]$ is isomorphic to the complete bipartite graph $K_{k,l}$. That is, whether there exists a 1-factorization $\mathcal{M} = \{M_1, \dots, M_k, N_1, \dots, N_l\}$ with any union $M_i \cup N_j$ forming a Hamilton cycle.

A 1-factorization $\mathcal{M} = \{M, N_1, \dots, N_{d-1}\}$ is called *semi-perfect* when $M \cup N_i$ is a Hamilton cycle for all i. Craft [Arc95] conjectured that for every integer $d \geq 2$ there is a semi-perfect 1-factorization of Q_d. This conjecture was proved independently by Gochev and Gotchev [GG10] and by Královič and Královič [KK05] in the case where d is odd, and settled for d even by Chitra and Muthusamy [CM13].

Gochev and Gotchev further proved that some other complete bipartite graphs are attainable. In particular they proved that when k and d are both even then Q_d has a 1-factorization $\mathcal{M}$ where $G[\mathcal{M}] = K_{k,d-k}$. In my work I resolve the question almost completely, with only one case missing.

Theorem 3.2. *[Beh19] For $k, l \in \mathbb{N}$ not both equal to 3, there is a 1-factorization $\mathcal{M}$ of the hypercube Q_{k+l} such that $G[\mathcal{M}]$ is isomorphic to the complete bipartite graph $K_{k,l}$.*

The idea of the proof is to construct such a 1-factorization. To do so, we use the product structure of $Q_{k+l} = Q_k \times Q_l$ combined with a result due to Stong [Sto06] which says that the symmetric directed hypercube $\overleftrightarrow{Q_d}$ can be partitioned into d directed Hamilton cycles when $d \neq 3$. The idea is to turn these directed Hamilton cycles into matchings and carefully assign copies of different matchings to the edges of Q_{k+l} in such a way that they are all disjoint and certain pairs of them always knit together to form a Hamilton cycle.

When one of k or l is 3 we cannot apply Stong's result and we must deal with these cases separately, requiring another two specialised constructions.

4 Connectivity of high dimension k-nearest neighbour graphs

Suppose that you had n radios in d-dimensional space and each radio can communicate with its k nearest neighbours. How large does k need to be, in relation to n and d, for every radio to be able to communicate (maybe indirectly) with any other? (i.e. for the graph to be connected?)

We formalise this notion by defining a graph $G = G(d, n, k)$ as follows: Let n and k be positive integers. Take a d-dimensional cube of volume n and let $\mathcal{P}$ be a Poisson process of density 1 in the cube. Join every point of $\mathcal{P}$ with its k nearest neighbours (in the Euclidean metric). In the same way we can define a directed graph $\overrightarrow{G} = \overrightarrow{G}(d, n, k)$ where every point of $\mathcal{P}$ has an directed edge out to each of its k nearest neighbours.

When the dimension d is fixed there exists a constants c such that for any $\epsilon > 0$, if $\frac{k}{\log n} < c - \epsilon$ then G is disconnected with high probability (probability tending to 1 as $n \to \infty$), and if $\frac{k}{\log n} > c + \epsilon$ then G is connected with high probability. (For proof see [Bal+05], for example - here it is done for the 2-dimensional case but the arguments generalise naturally.)

The question then becomes, can we say something about how this constant depends on the dimension d? Using very simple generalisations of arguments in [Bal+05] it is easy to show that we can get a lower bound of $O\left(\frac{1}{e^d}\right)\log n$ and an upper bound of $O\left(\frac{1}{\log d}\right)\log n$. I managed to improve these bounds considerably to a lower bound of $O\left(\frac{1}{d}\right)\log n$ and an upper bound of $O\left(\frac{1}{d \log d}\right)\log n$.

We first show that with high probability there cannot be two components of diameter greater than $c(\log n)^{\frac{1}{d}}$ and so it is enough to consider small diameter components. We can bound the probabilities of small diameter component for both the undirected and directed cases, as shown in Table 1.

	Small component without adjacent		
	edges	out-edges	in-edges
No such component if $\frac{k}{\log n} >$	$\frac{2.467}{d}$	$\frac{1}{\log d}$	$\frac{1}{d}2^d$
Exists such component if $\frac{k}{\log n} <$	$\frac{0.102}{d \log d}$	$\frac{0.721}{d}$	$\frac{0.079}{\log d}$

Table 1: Bounds on the thresholds for the existence of small diameter components.

Each one of these bounds involves a careful construction that takes as a start point those in [Bal+05] and [Wal12]. These had to be first adapted to be generalisable to higher dimensions and then further optimised (for example by varying the densities and the distances used). On particular issue is that in higher dimensions as a larger proportion of the box is near the boundary and so boundary effects cannot be waved away as in the two-dimensional case.

5 Directions for further work

For each of the problems I have worked on so far there are still open questions or conjectures which are discussed in more detail above.

I am also keen to apply my techniques and knowledge to problems that are related but not necessarily direct descendants of the above problems. Natasha Morrison and Jonathon Noel studied saturation in the hypercube, showing specifically in [MNS17] that the minimum number of edges in a (Q_d, Q_m)-saturated graph is $\Theta(2^d)$. There are several interesting open questions that follow on from their work that would be interesting to explore further, including saturation numbers for finite grids or even saturation density inside infinite square lattices.

Further, Morrison and Noel have studied bootstrap percolation, both in the hypercube [MNS17; MN18b] and in random hypergraphs [MN18a]. For the former in particular, they exploit the connection between bootstrap percolation and weak saturation. Bootstrap percolation is closely related not only to weak saturation but also to cellular automata, which are a special subclass of deterministic automata. I would be interested in exploring bootstrap percolation for other natural families of graph such as the d-dimensional torus.

One potential rich vein of bootstrap percolation questions to collaborate on is probabilistic problems. The biggest open question in this area is determining the density at which a random infection becomes likely to percolate: the so-called critical density. Morrison and Noel's result on bootstrap percolation in the hypercube was in fact originally motivated by the problem of determining the asymptotics of the critical density for r-neighbour bootstrap percolation in the hypercube.

Solving the critical density problem is an ambitious project; a more modest goal might be to investigate the critical probability for a bond percolation anaolgue of the r-neighbour bootstrap process. Specifically, this is a model where each edge is either healthy or infected and a healthy edge becomes infected if one of its endpoints is incident to at least r infected edges. This is again connected to weak saturation: the infection percolates under this rule if and only if the graph of infected edges is weakly saturated with respect to the star with $r + 1$ leaves. The critical probability for this process is unknown for hypercubes and for grids, which may well be fruitful areas to focus on.

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